Finite difference approximations to one-dimensional parabolic equations using a cubic spline technique

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ABSTRACT

This paper uses a cubic spline approximation to produce finite difference representations of the homogeneous heat equation in one spatial variable; it is shown that the usual explicit and implicit formulae are particular cases of the formulations given here. Formulae for truncation error and conditions for stability are derived. Numerical results are given for a simple example.

1. INTRODUCTION

The cubic spline is now a well established tool for the solution of two-point boundary value problems (Bickley [1]; Albasiny and Hoskins [2]; Fyfe [3]) and integral equations (Phillips [4]; Sastry [5]; Netravali and Figueiredo [6]). More recently, spline functions were utilized for the construction of spatial collocation in partial differential equations (See Douglas and Dupont [7]; Raggett and Wilson [8]). In [8], a cubic spline approximation to the one-dimensional wave equation was developed and this method is adopted here for the solution of parabolic partial differential equations.

In the sequel, we discuss the numerical solution of the homogeneous heat equation in one spatial variable and derive two classes of formulae involving two-level and three-level finite differences which reduce to the frequently used formulae of Crank-Nicolson and Douglas, and of Du Fort-Frankel.

An account of the basic properties of the cubic spline can be found in Ahlberg, Nilson and Walsh [9], but for the purpose of this note it is sufficient to state that the cubic spline $S(x)$ interpolating to the function $y(x)$ at the knots

$x_j = x_o + jh$ $(j = 0, 1, 2, \ldots, n)$ is given in the interval $x_{j-1} \leqslant x \leqslant x_j$ by the equation

$$\begin{aligned} S(x) &= M_{j-1}\frac{(x_j-x)^3}{6h} + M_j\frac{(x-x_{j-1})^3}{6h} \\ &+ \left(y_{j-1}-\frac{h^2}{6}M_{j-1}\right)\frac{(x_j-x)}{h} \\ &+ \left(y_j-\frac{h^2}{6}M_j\right)\frac{(x-x_{j-1})}{h} \end{aligned} \tag{1.1}$$

where $M_j = S''(x_j)$ and $y_j = y(x_j)$. The condition of continuity of the first derivatives implies

$$M_{j-1} + 4M_j + M_{j+1} = \frac{6}{h^2}(y_{j-1} - 2y_j + y_{j+1})$$
$$(j = 1, 2, \ldots, n-1) \tag{1.2}$$

The purpose of this note is to derive finite difference representations for the solution of the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.3}$$

subject to the initial condition

$$u = g(x) \quad \text{at} \quad t = 0, \quad \text{for } 0 \leqslant x \leqslant \pi \tag{1.4}$$

and the boundary conditions

$$\left.\begin{aligned} u &= f_1(t) \quad \text{at } x = 0 \\ \text{and } u &= f_2(t) \quad \text{at } x = \pi, \quad \text{for } t > 0 \end{aligned}\right\} \tag{1.5}$$

where $g(x)$, $f_1(t)$ and $f_2(t)$ are known functions.

In § 2, the finite difference representations and their stability criteria are derived and in § 3 the computational procedure is described and a numerical example given.

2. FINITE DIFFERENCE REPRESENTATIONS

A simple explicit approximation to (1.3) is

$$u_{i,j+1} - u_{i,j} = r\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}], \tag{2.1}$$

where $r = k/h^2$, $u_{i,j}$ denoting the value of u at the mesh point (ih, jk), $i = 0, 1, 2, \ldots, n$; $j = 1, 2, \ldots,$ and $nh = \pi$.

A more general implicit scheme, due to Crank and

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Nicolson, is given by

$$(1+r)\, u_{i,j+1} - \frac{1}{2} r\,(u_{i+1,j+1} + u_{i-1,j+1})$$

$$= (1-r)\, u_{i,j} + \frac{1}{2} r\,(u_{i+1,j} + u_{i-1,j}), \qquad (2.2)$$

which is unconditionally stable.
The local truncation errors associated with the schemes (2.1) and (2.2) are respectively given by (see Fox [10] page 238) :

$$\left(\frac{1}{2} k - \frac{1}{12} h^2\right) \frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{6} k^2 - \frac{1}{360} h^4\right) \frac{\partial^3 u}{\partial t^3} + \ldots \qquad (2.3)$$

and

$$-\frac{1}{12} h^2 \frac{\partial^2 u}{\partial t^2} - \left(\frac{1}{12} k^2 + \frac{1}{360} h^4 + \frac{1}{24} k h^2\right) \frac{\partial^3 u}{\partial t^3} + \ldots \qquad (2.4)$$

A formula of maximum accuracy based on the same six mesh points as the Crank-Nicolson formula (2.2) is that due to Douglas (See, Mitchell [11]) :

$$u_{i,j+1} - u_{i,j} = \frac{1}{2}\left(r + \frac{1}{6}\right)[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

$$+ \frac{1}{2}\left(r - \frac{1}{6}\right)[u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}], \qquad (2.5)$$

with a local truncation error of $0\,[k^2 + h^4]$.
A new finite difference approximation to (1.3) is now obtained by replacing the time derivative $\frac{\partial u}{\partial t}$ by the finite difference approximation $(u_{i,j+1} - u_{i,j})/k$ and the space derivative $\frac{\partial^2 u}{\partial x^2}$ by a cubic spline. If $S_j(x)$ denotes the cubic spline approximating the function values $u_{i,j}$, then a suitable approximation to (1.3) is given by

$$\frac{1}{k}[u_{i,j+1} - u_{i,j}] = (1-\theta) M_{i,j} + \theta M_{i,j+1},$$

$$i = 0, 1, 2, \ldots, n;\ j = 1, 2, \ldots \qquad (2.6)$$

where $M_{i,j} = S_j''(x_i)$.

Analogous with equation (1.2) the following relations hold :

$$M_{i-1,j} + 4M_{i,j} + M_{i+1,j} =$$

$$= \frac{6}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}], \qquad (2.7)$$

and

$$M_{i-1,j+1} + 4M_{i,j+1} + M_{i+1,j+1} =$$

$$= \frac{6}{h^2}[u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}] \qquad (2.8)$$

Relations (2.6) through (2.8) yield the following finite difference formula

$$(1-6r\theta)(u_{i-1,j+1} + u_{i+1,j+1}) + (4+12r\theta) u_{i,j+1}$$

$$= [1+6r(1-\theta)](u_{i-1,j} + u_{i+1,j}) + [4-12r(1-\theta)] u_{i,j}$$

$$(i = 1, 2, \ldots, n-1) \qquad (2.9)$$

where $r = k/h^2$ as before. The formula (2.9) is a general implicit representation to (1.3) and reduces to the approximations (2.1), (2.2) and (2.5) by particular choices of the parameter θ. Thus the choice of $\theta = \frac{1}{6r}$ gives rise to (2.1), $\theta = \frac{1}{2} + \frac{1}{6r}$ leads to (2.2), whilst (2.5) is obtained by choosing $\theta = \frac{1}{2} + \frac{1}{12r}$.

The truncation error associated with the scheme (2.9) is given by

$$\left[\frac{1}{12} h^2 + \frac{1}{2} k(1-2\theta)\right] \frac{\partial^2 u}{\partial t^2} + \left[\frac{1}{90} h^4 + \frac{1}{6} k^2 (1-3\theta)\right.$$

$$\left. + \frac{1}{12} h^2 k (1-\theta)\right] \frac{\partial^3 u}{\partial t^3} + \ldots \qquad (2.10)$$

It is easy to verify that this expression reduces to (2.3) for $\theta = \frac{1}{6r}$ and to (2.4) for $\theta = \frac{1}{2} + \frac{1}{6r}$.

Further, it can be shown that the scheme (2.9) is stable when θ satisfies the condition :

$$1 - 2\theta \leqslant \frac{1}{6r}, \qquad (2.11)$$

from which follows the condition of stability of the explicit scheme (2.1) by setting $\theta = \frac{1}{6r}$.

A three-level difference scheme for the solution of (1.3) is obtained by starting with the approximation

$$\frac{1}{2k}[u_{i,j+1} - u_{i,j-1}] = \lambda M_{i,j-1} + (1-\lambda-\mu) M_{i,j} + \mu M_{i,j+1} \qquad (2.12)$$

Using a procedure similar to the one outlined above, the following finite difference formula is obtained :

$$(1-12r\mu)(u_{i-1,j+1} + u_{i+1,j+1}) + (4+24r\mu) u_{i,j+1}$$

$$= (1+12r\lambda)(u_{i-1,j-1} + u_{i+1,j-1})$$

$$+ (4-24r\lambda) u_{i,j-1}$$

$$+ 12r(1-\lambda-\mu)(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \qquad (2.13)$$

Setting $\mu = -\lambda = \frac{1}{12r}$, there results

$$\frac{1}{2k}[u_{i,j+1}-u_{i,j-1}]$$

$$=\frac{1}{h^2}[u_{i-1,j}-2u_{i,j}+u_{i+1,j}], \qquad (2.14)$$

which is the simplest explicit scheme, and is unconditionally unstable.

Replacing $u_{i,j}$ by $\frac{1}{2}[u_{i,j-1}+u_{i,j+1}]$ in (2.13), the following scheme is obtained :

$$\begin{aligned}&(1-12r\mu)(u_{i-1,j+1}+u_{i+1,j+1})\\&+[4+24r\mu+12r(1-\lambda-\mu)]u_{i,j+1}\\&=(1+12r\lambda)(u_{i-1,j-1}+u_{i+1,j-1})\\&+[4-24r\lambda-12r(1-\lambda-\mu)]u_{i,j-1}\\&+12r(1-\lambda-\mu)(u_{i-1,j}+u_{i+1,j}),\end{aligned} \qquad (2.15)$$

where the local truncation error is

$$\begin{aligned}&(48rk\mu-4k)\frac{\partial u}{\partial t}+[12k^2(\lambda+\mu)\\&+\frac{12k^3}{h^2}-\frac{12k^3}{h^2}(\lambda+\mu)-kh^2]\frac{\partial^2 u}{\partial t^2}\\&+[\frac{8k^4}{h^2}\mu-6k^3(\lambda+\mu)+k^2h^2(\lambda+\mu)\\&-\frac{kh^4}{30}+\frac{4}{3}k^3]\frac{\partial^3 u}{\partial t^3}+\dots\end{aligned} \qquad (2.16)$$

Formula (2.15) is a three-level finite difference approximation for the solution of equation (1.3) and it reduces to the Du Fort-Frankel scheme for $\mu=-\lambda=\frac{1}{12r}$:

$$(1+2r)u_{i,j+1}=(1-2r)u_{i,j-1}+2r(u_{i-1,j}+u_{i+1,j}) \qquad (2.17)$$

3. COMPUTATIONAL PROCEDURE

The cubic spline solution can now be obtained numerically by the following computational procedure.

Utilizing formulae (2.9) or (2.15) requires the solution of a tridiagonal linear system at each time step; subsequently the values of M_i in (1.1) for each point of time can be obtained by solving the tridiagonal linear system consisting of (1.2) and the two supplementary conditions

$$M_o=f_1'(t) \text{ and } M_n=f_2'(t), \qquad (3.1)$$

resulting from the collocation identities

$$\frac{\partial}{\partial t}S(x_i,t)=\frac{\partial^2}{\partial x^2}S(x_i,t)=M_i(t), \qquad (3.2)$$

and the natural requirements

$$S(x_o,t)=f_1(t) \text{ and } S(x_n,t)=f_2(t) \qquad (3.3)$$

To illustrate the procedure outlined above, the equation (1.3) is solved (as in Mitchell [11], page 26) with the initial condition $u=\sin x$ $(0\leqslant x\leqslant\pi)$ at $t=0$ and the boundary conditions $u=0$ at $x=0,\pi$ $(t\geqslant 0)$. The formulae (2.9) and (2.15) are used choosing in each case $h=\pi/20$ and $r=1/\sqrt{20}$.

The computations are made for different values of θ on an IBM 360 model 44 computer using the double precision arithmetic, and it is found that the values obtained with $\theta=\frac{1}{2}+\frac{1}{6r}$ and $\theta=\frac{1}{2}+\frac{1}{6r}$ (corresponding respectively to the Crank-Nicolson and Douglas schemes) compare favourably (to within small differences) with those tabulated in [11]. The method has the obvious advantage that it produces a spline function valid on each new time line which can be used to obtain the solution at any point in the range, whereas the finite difference formulae given by (2.1), (2.2), (2.5) and (2.17) obtain the solution only at the chosen knots.

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